The Sorrows of a Young Algebraist

Tomáš Nagy

Jagiellonian University

joint work with
J. B. Brunar, J. W. Goethe, M. Kozik, and M. Pinsker

AAA 108, Wien, 8th February 2026

Neither the author nor the co-authors are responsible
for any consequences or side effects of this talk.

Charlotte: CSP
(Johanna Beate Brunar)

Cast

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Werther (aka the Young Algebraist): Universal Algebra
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Albert (Charlotte's fiancé and later husband): Computer Science
(any volunteers?)

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$\rightsquigarrow$ the Bulatov-Zhuk dichotomy theorem relies on algebraic machinery

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New algebraic theory developed recently
(smooth approximations; Mottet, Pinsker, 2022)

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$\Rightarrow$ Most of the finite-domain methods cannot be lifted

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More to come? Pseudo-identities, local consistency, ...

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Barto-Kozik-Niven.

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$\mathbb{G}$ – smooth ω-categorical digraph

6-ary pseudo-Siggers identity (Barto, Pinsker, 2016)

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Charlotte's Dark Secrets

Pseudo-Identities and Pseudo-Loop Conditions

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For $\mathbb{G}$ finite proven by Bodor, Kozik, Mottet, Pinsker, 2023

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In the infinite: Vertices of $\mathbb{G} \rightsquigarrow$ orbits of elements under $\mathrm{Aut}(\mathbb{G})$

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Hope: Techniques might generalise further

- Further de-algebraising of the finite proofs

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  - Connection between identities and PCSPs (Mottet, 2025)

Figure: Thank you for your attention!