Neoliberalism and local consistency

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joint work with Michael Pinsker

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Infinite structures

$\mathbb{B}$ *homogeneous* if every orbit under $\mathrm{Aut}(\mathbb{B})$ determined by relations

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Example: $(\mathbb{Q}; <)$ does NOT have finite duality:
all cycles forbidden $x_1 < x_2 < \cdots < x_n < x_1$.

the universal homogeneous triangle-free graph has finite duality

Infinite-domain CSPs

$\mathbb{B}$ - finitely bounded, homogeneous
$\mathbb{A}$ - first-order definable in $\mathbb{B}$

$\mathrm{CSP}(\mathbb{A})$:

Input: $\Phi = \phi_1 \wedge \ldots \wedge \phi_k$ - conjunction of atomic formulas over the signature of $\mathbb{A}$
Question: Φ satisfiable?

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Finite formulation:
$\mathrm{maxarity}(\mathbb{B}) = k$, τ - signature of $\mathbb{B}$

Given:

- "values": $O_1, \ldots, O_m$ - k-orbits under $\mathrm{Aut}(\mathbb{B})$,
- "constraints": constraints given by Φ (quantifier-free τ-formulas) + $\{F_1, \ldots, F_n\}$ - finite forbidden τ-structures (bounds)

Want: assign to every k-tuple of free variables of Φ an orbit O_i s.t. no F_i embeds to the resulting structure and s.t. Φ is satisfied

Liberalism vs neoliberalism, 1/2

$\mathbb{B}$ is *liberal* if its relations correspond to orbits of pairs
and it does not have bounds of size $3-6$
"If you are not free, you at least do not notice it."

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  - $\rightsquigarrow$ every orbit determined by k-ary relations
  - *clear and concise regulations*

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  - *it is easy to divert money and avoid taxes*

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liberal $\Rightarrow$ 2-neoliberal

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Examples:

- $(\mathbb{Q}; <, =)$ is 2-neoliberal but not liberal
  - orbits determined by $<, =$,
  - any $a \in \mathbb{Q}$ can be moved by an automorphism to any other $b \in \mathbb{Q}$ $\Rightarrow$ one orbit of elements,
  - for any $a \in \mathbb{Q}$, there exist $b \neq c \in \mathbb{Q}$ with $a < b, a < c$,
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- graph $\mathbb{G}$ consisting of infinitely many isolated edges is NOT 2-neoliberal
  - for any $a \in G$, there is a unique b connected by an edge to a
    - $\Rightarrow$ impossible to divert money

$\Phi = \phi_1 \wedge \cdots \wedge \phi_k$ – instance of $\mathrm{CSP}(\mathbb{A})$
How to solve $\mathrm{CSP}(\mathbb{A})$?

Local consistency: Derive information locally,
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"Example": Computing the transitive closure of a binary relation R.
$\phi_i : R(x, y)$, $\phi_j : R(y, z) \Rightarrow$ add $\phi := R(x, z)$ to Φ

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$R^{\mathbb{A}}$ irreflexive, transitive and we derive $R(x, x) \Rightarrow \Phi$ not satisfiable.

$\Rightarrow$ sometimes, local consistency *solves* $\mathrm{CSP}(\mathbb{A})$

Local consistency, 2/4

$1 \leq m \leq n$
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scope S of ϕ_i: all variables of ϕ_i

projection of ϕ_i to $X \subseteq S$: $\exists x_1 \ldots x_\ell \phi_i$, where $S \backslash X = \{x_1, \ldots, \ldots x_\ell\}$

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- for every set of $\leq n$ variables from $\mathcal{V}$, some ϕ_i contains all these variables in its scope, and
- for every set V of $\leq m$ variables from $\mathcal{V}$ and for all ϕ_i, ϕ_j containing all variables from V in their scopes, the projections to V agree.

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$\mathbb{A}$ has *(relational) width* (m, n) if every non-trivial (m, n)-minimal instance satisfiable

$\rightsquigarrow$ local consistency solves $\mathrm{CSP}(\mathbb{A})$

Local consistency, 3/4

Examples:

- $(\mathbb{Q}; =, <)$ has width $(2, 3)$
  - Idea: ensure that the transitive closure of $<$ is irreflexive.
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$\Rightarrow$ Grand coalition (“building bridges”)

Local consistency, 4/4

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Collapse (Barto, 2016)
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Question: $\mathbb{A}$ fo-definable in a finitely bounded homogeneous $\mathbb{B}$,
$\mathbb{A}$ has bounded width.
Does there exist a bound on the width of $\mathbb{A}$ *depending only on* $\mathbb{B}$*?*

Bounds on width, 1/2

$\mathbb{A}$ fo-definable in $\mathbb{B}$
$k - maxarity(\mathbb{B})$, ℓ – size of the biggest bound

Does there exist a bound on the width of $\mathbb{A}$ depending only on $\mathbb{B}$?

Assume: $\mathbb{A}$ has a relation for every orbit of k-tuples under $\mathrm{Aut}(\mathbb{B})$.

What is the minimal possible width of $\mathbb{A}$?

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- Need $($something$, \ell)$ to get all constraints given by bounds.
- If $=$ among relations of $\mathbb{A} \Rightarrow$ need $(k, k+1)$ to exclude

$$(x_1, \dots, x_k) \in O, (x_1, \dots, x_{k-1}, y) \in P, x_k = y$$

  for $O \neq P$

$\rightsquigarrow$ $\mathbb{A}$ has relational width at least $(k, \max(k+1, \ell))$.

Bounds on width, 2/2

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Idea: $k = 1$, $\ell = 2$ (forbid $a \in \{i\} \cap \{j\}$, $a, b \in \{i\}$)
$\Rightarrow$ Natural guess for upped bound on the width of $\mathbb{A}$: $(2k, \max(3k, \ell))$

Is this true also for infinite $\mathbb{A}$???

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Often YES.
No counterexample known!

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$m \geq 1$

$\mathbb{A}$ has *strict width* m if there exists $n \geq m$
s. t. for every (m, n)-minimal instance,
any *local solution* can be extended to a global one.

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Algebraic characterization: finite or infinite (ω-cat.) $\mathbb{A}$ has strict width k $\Leftrightarrow$ for every finite $F \subseteq A$,
$\exists$ a $(k+1)$-ary *polymorphism* of $\mathbb{A}$ which is a *near-unanimity* on F:
$x \approx f(x, \ldots, x) \approx f(y, x, \ldots, x) \approx \cdots \approx f(x, \ldots, x, y)$

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$m \geq 1$

$\mathbb{A}$ has *strict width* m if there exists $n \geq m$
s. t. for every (m, n)-minimal instance,
any *local solution* can be extended to a global one.

$\Phi = \phi_1 \wedge \cdots \wedge \phi_k$ – instance of $\mathrm{CSP}(\mathbb{A})$ over variables $\mathcal{V}$
Want: for any $U \subseteq \mathcal{V}$, any assignment $f : U \to A$ satisfying projection of every ϕ_i to U can be extended to a satisfying assignment for Φ.

$\Rightarrow$ far-right (extreme local consistency, controls too much,
kills everybody who doesn't contribute to the intended global solution)

Example: the universal triangle-free graph has strict width 2
(need $(2, 3)$-minimality)

Algebraic characterization: finite or infinite (ω-cat.) $\mathbb{A}$
has strict width $k \Leftrightarrow$ for every finite $F \subseteq A$,
$\exists$ a $(k+1)$-ary *polymorphism* of $\mathbb{A}$ which is a *near-unanimity* on F:
$x \approx f(x, \ldots, x) \approx f(y, x, \ldots, x) \approx \cdots \approx f(x, \ldots, x, y)$

No collapse even for finite $\mathbb{A}$!

A contribution to the progress of the human race

$k \geq 3$,
$\mathbb{B}$ – k-neoliberal, has finite duality,
ℓ – size of the biggest bound for $\mathbb{B}$
$\mathbb{A}$ – fo-definable in $\mathbb{B}$, has all relations of $\mathbb{B}$

Theorem. [N., Pinsker]

If $\mathbb{A}$ has bounded strict width
$\Rightarrow$ $\mathbb{A}$ has relational width $(k, \max(k+1, \ell))$.

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Idea: using the algebraic characterization of strict width,
show that certain "implications" $R(x_1, \ldots, x_m) \Rightarrow S(y_1, \ldots, y_n)$
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"Neoliberalism implies that if a problem can be solved
by installing a fascist regime (strict width),
it can be solved in a much easier way and with less resources
using conservative policies (relational width)."


Thank you for your attention!