The Strength and Limitations of the Canonical Reduction for Infinite-Domain CSPs

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joint work with
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$\mathbb{A}$ – relational structure

Constraint Satisfaction Problem over $\mathbb{A}$ (CSP($\mathbb{A}$))

INPUT: $\mathbb{I}$ – relational structure over the same signature as $\mathbb{A}$

OUTPUT: Does there exist a homomorphism $h\colon \mathbb{I} \to \mathbb{A}$?

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$\mathrm{CSP}(\{0,1\};\{(x,y,z) \mid x+y+z=0\},\{(x,y,z) \mid x+y+z=1\})$ – linear equations over $\mathbb{Z}_2$

Finite-domain dichotomy

Theorem [Bulatov, Zhuk].

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Polymorphisms: Example

$\mathbb{A} := \mathrm{CSP}(\{0,1\}; R_0 = \{(x,y,z) \mid x+y+z=0\},$
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Works for any set of linear equations over a finite field

Observe: $x = m(x,x,x) = m(x,y,y) = m(y,y,x)$ for any x, y

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Fun fact: m is a *Mal'tsev polymorphism*,
any CSP with such a polymorphism can be solved
by a generalised Gaussian elimination

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- Specific structures:
  - graphs: has a loop, is bipartite, or is NP-c (Hell, Nešetřil),
  - digraphs (Barto, Kozik, Niven)
- Particular polymorphisms, e.g. Mal'tsev
  $\rightsquigarrow$ Bulatov-Dalmau algorithm

Infinite-domain?

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Fun subfact: The complexity still depends only on polymorphisms!

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Example: $\mathrm{CSP}(\mathbb{Q},<)$, $\mathbb{I} := (\{x,y,z\}; x<y, x<z)$.
Finitely many solutions up to Aut: $x<y<z$, $x<z<y$, $x<y=z$ ✓

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Homogeneity: Every solution described by conjuction of $<$ and $=$ ✓

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Homogeneity: Every solution described by conjuction of $<$ and $=$ ✓
Finite boundedness: Forbidden formulas: $x < x$, $x < y \wedge y < x$, $x < y < z < x$, x, y unrelated ✓

The infinite-domain conjecture

Conjecture [Bodirsky-Pinsker].

Let $\mathbb{B}$ be first-order definable from a finitely-bounded homogeneous $\mathbb{A}$. Then $\mathrm{CSP}(\mathbb{B})$ is either tractable, or NP-complete.

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First-order definable $\leadsto$ solution space still algorithmically enumerable: Take the solution space for $\mathbb{A}$

Example: $\mathbb{A} := (\mathbb{Q}, <)$, $\mathbb{B} := (\mathbb{Q}, \{(x, y, z) \mid x < y < z \lor z < y < x\})$
$\leadsto$ Betweenness (NP-complete)

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Example: $\mathbb{A} := (\mathbb{Q}, <)$, $\mathbb{B} := (\mathbb{Q}, \{(x, y, z) \mid x < y < z \vee z < y < x\})$
$\rightsquigarrow$ Betweenness (NP-complete)

Confirmed when $\mathbb{A}$ is:

- unary,
- $(\mathbb{Q}, <)$,
- graph,
- certain special structures (phylogeny CSPs, Network CSPs, ...),
- certain hypergraphs and tournaments.

Solution spaces of infinite-domain CSPs

$\mathbb{A} := (\mathbb{Q}, <)$,
$\mathbb{B} := (\mathbb{Q}, \mathrm{Betw})$, where $\mathrm{Betw} := \{(x, y, z) \mid x < y < z \vee z < y < x\}$
$\mathbb{I} := (\{x, y, z, u\}; \mathrm{Betw}(x, y, z), \mathrm{Betw}(u, x, z))$ – instance of $\mathrm{CSP}(\mathbb{B})$

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Another way of viewing this solution:
$(x, y) \mapsto <$, $(x, z) \mapsto <$, $(x, u) \mapsto >$, $(y, z) \mapsto <$, $(y, u) \mapsto >$, $(z, u) \mapsto >$

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$\rightsquigarrow$ instance of a finite-domain CSP over a 3-element domain $<, >, =$

But: Not every instance of this finite-domain CSP comes from an instance of $\mathbb{B}$!

When this reduction doesn't work

Does this reduction always preserve tractability?:
Take $R := \{(x, y, z) \in \mathbb{Q}^3 \mid x \neq y \neq z \neq x\}$.
$\rightsquigarrow \mathrm{CSP}(\mathbb{Q}; R)$ obviously tractable:
Any instance without $R(x, x, .)$ is a YES instance, as we can always find a strict linear order on a finite set.

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But: possibilities for relations on $(x, y), (y, z), (z, x)$

(x, y)	(y, z)	(z, x)
$<$	$<$	$>$
$<$	$>$	$<$
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$< \mapsto 0$
$> \mapsto 1$

(x, y)	(y, z)	(z, x)
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$\rightsquigarrow$ NAE-SAT (all triples over $0,1$ where not all entries are equal) – NP-complete

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Canonical: Preserves the equivalence modulo automorphisms, i.e., act on the finite solution space:

For $(\mathbb{Q}, <)$: Expressions like $f(<, >, =) =<$ make sense, meaning:

For all $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{Q}^3$ with $a_1 < a_2, b_1 > b_2, c_1 = c_2$, it holds that $f(a_1, b_1, c_1) < f(a_2, b_2, c_2)$

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$\mathbb{G} = (V; E)$ – the *Rado graph*, subgraphs are all finite loopless graphs

Solution space described by: E (edge), N (non-edge), $=$

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Obtain R_1 from R_0 by switching E's and N's

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Obtain R_1 from R_0 by switching E's and N's

Observe: Understand E as 1 and N as 0
$\rightsquigarrow \mathbb{B} := (V; R_0, R_1)$ is basically the same as $(\{0,1\}; R_0, R_1)$
(formally, we reduce to a CSP over the domain $E, N, =$)

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Obtain R_1 from R_0 by switching E's and N's

Observe: Understand E as 1 and N as 0
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$\rightsquigarrow$ Reduce $\mathrm{CSP}(\mathbb{B})$ to linear equations over $\mathbb{Z}_2$,
the polymorphism m witnesses tractability

Back to the finite dichotomy

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$(2, 2)$ not enough (would never derive $x < z$)

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Collapse of the bounded width hierarchy

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$\rightsquigarrow$ in general, no collapse possible

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$\mathbb{A}$ – finitely bounded, homogeneous,
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- unary structures, ...

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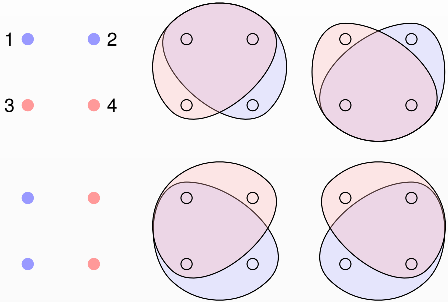
Mottet, Pinsker: Counterexample first-order definable in the random 3-uniform hypergraph!



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A novel algorithm needed to perform this reduction
in this infinite setting $\rightsquigarrow$

Theorem [Mottet, N., Pinsker].

Let $k \geq 1$ and $\mathbb{B}$ be a *resonable* k-uniform finitely-bounded hypergraph.
For any $\mathbb{A}$ first-order definable in $\mathbb{B}$,
its CSP is either tractable, or NP-complete.

What to do further with the B-P conjecture

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Need more examples to understand what can go wrong…